Applying Data Assimilation to Parabolic PDEs

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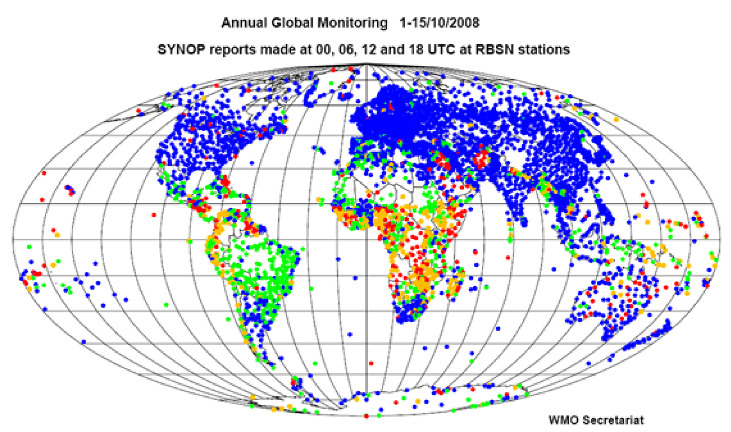
Introduction

Data Assimilation in Atmospheric Science:

- Weather prediction requires information of atmospheric conditions e.g. wind, humidity, temperature, etc. at an initial forecasting time
- Data assimilation combines mathematical models and limited number of observations to estimate atmospheric conditions at an initial forecasting time and predict unknown atmospheric conditions

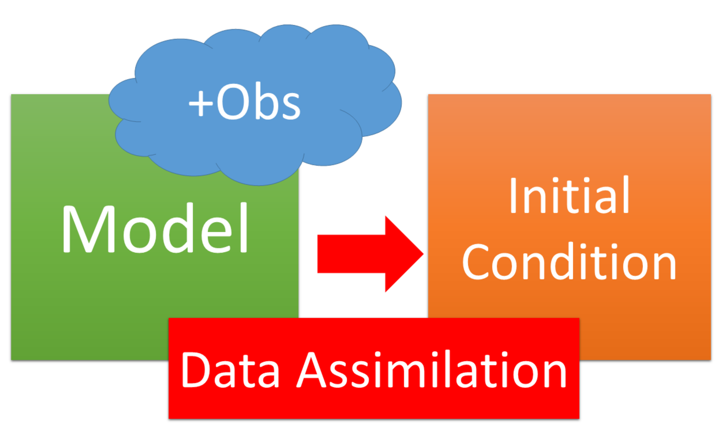
Challenges:

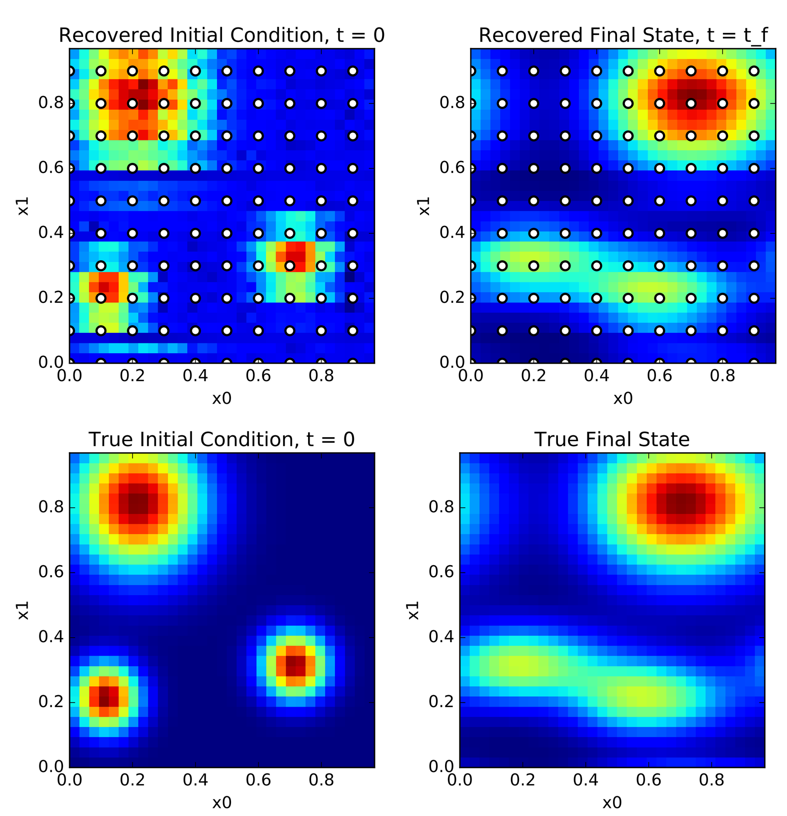
- Atmospheric conditions can only be observed at few locations in space
- The true state of the atmospheric conditions everywhere is unknown



Motivation

- Explore numerical methods for data assimilation by using simple mathematical models
- Determine how the analysis is influenced by length of time window and number of observation points





Data Assimilation Formulated as Least Squares

Model of physics $\{\mathbf{x}_k\}_{k=0}^{n_t} \subset \mathbb{R}^{n_x}$ (e.g., $\mathbf{x}_k$ = atmospheric condition at time t_k) obeys

$$\mathbf{x}_{k+1} = \mathbf{F}_k(\mathbf{x}_k) + \boldsymbol{\xi}_{k+1}, \qquad \boldsymbol{\xi}_k \sim N(\mathbf{0}, \boldsymbol{\Sigma}).$$

Observations $\{\mathbf{y}_k\}_{k=0}^{n_t}$ sample the state with $\mathbf{H} \in \mathbb{R}^{n_s \times n_x}$ as rows of the identity,

$$\mathbf{y}_k = \mathbf{H}\mathbf{x}_k + \boldsymbol{\zeta}_k, \qquad \boldsymbol{\zeta}_k \sim N(\mathbf{0}, \boldsymbol{\Gamma}).$$

Then the least squares formulation of the data assimilation problem is written,

$$\min_{\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_{n_t}} \frac{1}{2} \left\| \begin{matrix} \mathbf{H}\mathbf{x}_0 - \mathbf{y}_0 \\ \mathbf{F}_0(\mathbf{x}_0) - \mathbf{x}_1 \\ \mathbf{H}\mathbf{x}_1 - \mathbf{y}_1 \\ \vdots \\ \mathbf{F}_{n_t}(\mathbf{x}_{n_t-1}) - \mathbf{x}_{n_t} \\ \mathbf{H}\mathbf{x}_{n_t} - \mathbf{y}_{n_t} \end{matrix} \right\|^2_{\mathbf{W}^{-1}} \qquad \mathbf{W} = \begin{pmatrix} \boldsymbol{\Gamma} & & & & & \\ & \boldsymbol{\Sigma} & & & & \\ & & \boldsymbol{\Gamma} & & & \\ & & & \ddots & & \\ & & & & \boldsymbol{\Sigma} & \\ & & & & & \boldsymbol{\Gamma} \end{pmatrix},$$

where the least squares problem is weighted by the inverses of the covariances.

Data Assimilation Results

Parabolic Model (2D diffusion-advection-reaction with periodic boundary conditions):

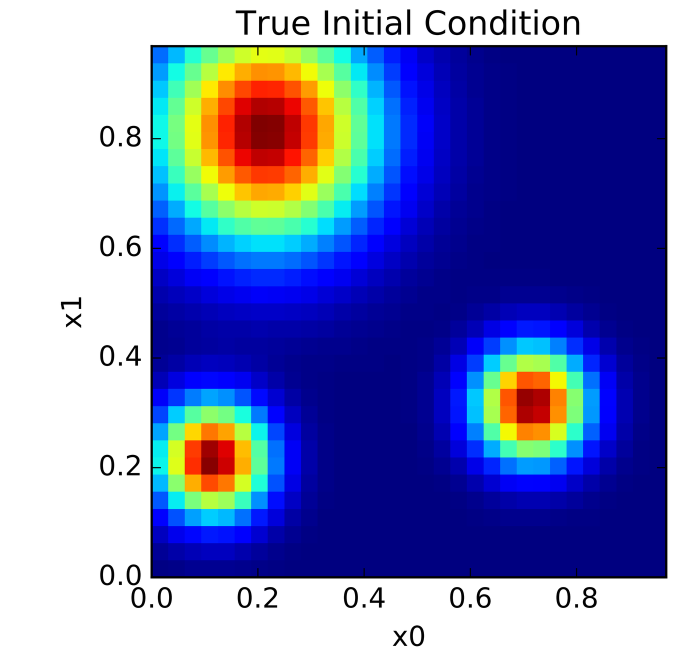
$$\frac{\partial}{\partial t} u(x_1, x_2, t) - \nu \Delta u(x_1, x_2, t) + \mathbf{a}^T \nabla u(x_1, x_2, t) + r u(x_1, x_2, t) = f(x_1, x_2, t), \qquad (x_1, x_2, t) \in (0,1)^2 \times (0, T)$$

$$u(x_1, x_2, 0) = u_0(x_1, x_2), \qquad (x_1, x_2) \in (0,1)^2$$

$$u(x_1, 0, t) = u(x_1, 1, t), \qquad \frac{\partial}{\partial x_1} u(x_1, 0, t) = \frac{\partial}{\partial x_1} u(x_1, 1, t), \qquad (x_1, t) \in (0,1) \times (0, T)$$

$$u(0, x_2, t) = u(1, x_2, t), \qquad \frac{\partial}{\partial x_2} u(0, x_2, t) = \frac{\partial}{\partial x_2} u(1, x_2, t), \qquad (x_2, t) \in (0,1) \times (0, T)$$

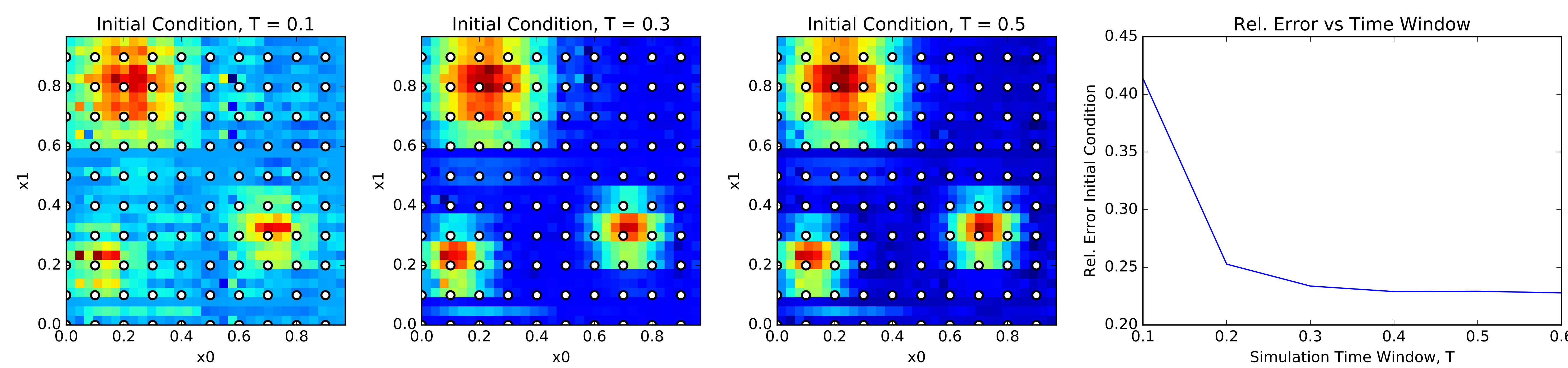
with coefficients $\nu = 0.001$, $\mathbf{a} = [1, 0]^T \in \mathbb{R}^2$, $r = 0$, and f, u_0 are given functions.



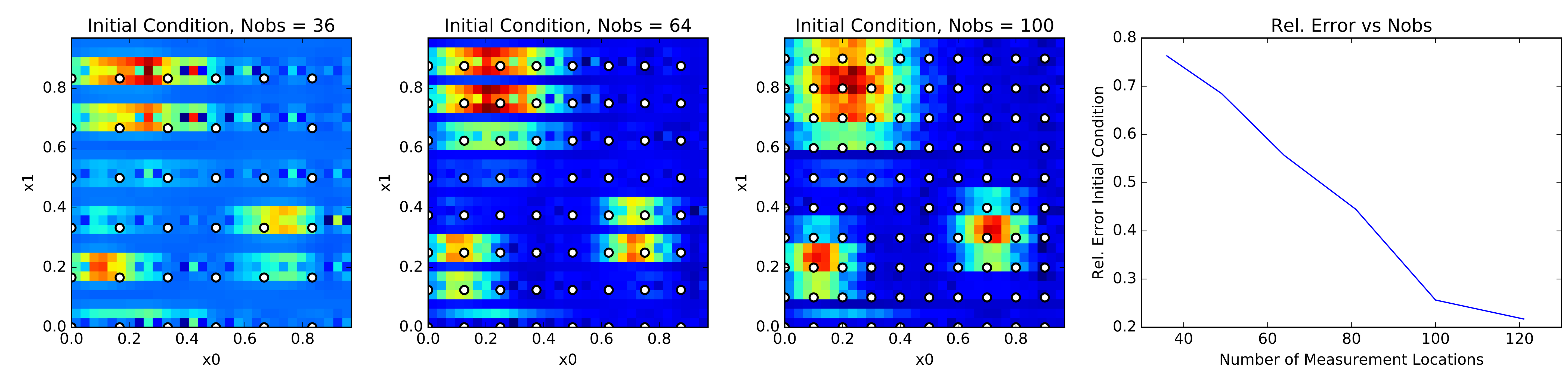
Experiment:

- ξ_k and ζ_k are Gaussian IID and $\boldsymbol{\Sigma} = \boldsymbol{\Gamma} = \sigma^2 \mathbf{I}$ where $\sigma = 0.001$ (0.1% noise)
- Square mesh, $[0,1]^2$ with Finite Difference discretization, $N_x = (32, 32)$, upwind stencil for advection

Impact of Observation Time Window on Estimation of Initial Condition:



Impact of Number of Observation Points on Estimation of Initial Condition:



Conclusions

- Data assimilation can be formulated as a least squares problem.
- Increasing time window and number of observation points improves quality of estimated initial condition.
- In this example, after some time adding measurement locations is more beneficial than running longer time windows.

Project Summary

- Python: Numerical library (NumPy+SciPy) and object oriented programming
- Least squares formulation (Conjugate Gradient and Gauss-Newton)
- Numerical methods to solve 1D and 2D parabolic PDEs
- Adjoint-based data assimilation for linear PDE model

References

1. Jeffrey. H, Preston.R, and Jeremy.W 2012: A Fresh Look at the Kalman Filter, *SIAM Rev.*, 54(4), 801-823.

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